ON THE SHORTEST LINE BETWEEN TWO POINTS IN NON-EUCLIDEAN GEOMETRY.

By T. H. Gronwall.

In his Science et hypothèse, Poincaré considers the geometry of a space interior to a sphere of radius R in which the line element equals $ds/(R^2 - r^2)$, where r is the distance from the center of the sphere and ds the ordinary euclidean line element. Poincaré states without proof (l. c., p. 65) that in this geometry the shortest line joining two given points is a circle through these points and orthogonal to the sphere. A simple proof of this theorem (which is of course well known in non-euclidean geometry) may be of some interest to the readers of the *Annals*.*

Using polar coördinates r, φ, θ, where the meridian plane $\varphi = 0$ passes through the two given points, we have $ds^2 = dr^2 + r^2 \sin^2 \theta d\varphi^2 + r^2 d\theta^2$, and the length of any curve $\varphi = \varphi(r)$, $\theta = \theta(r)$ is

$$\int \frac{\sqrt{1 + r^2 \sin^2 \theta \cdot \varphi'^2 + r^2\theta'^2}}{R^2 - r^2} dr,$$

the integral being taken between the limits r_1 and r_2 corresponding to the two given points. This integral exceeds or equals

$$\int \frac{\sqrt{1 + r^2\theta'^2}}{R^2 - r^2} dr \tag{1}$$

(which represents the length of the plane curve $\varphi = 0$, $\theta = \theta(r)$), equality taking place only when $\sin \theta \cdot \varphi' = 0$ for every point on the curve. Any point for which $\sin \theta = 0$ evidently lies in the meridian plane $\varphi = 0$, and $\varphi' = 0$ gives $\varphi = \text{const.} = 0$ (since $\varphi = 0$ at the two given points). Our shortest line is therefore a plane curve, and is found by minimizing the integral (1). The solution of the Euler-Lagrange equation

$$\frac{\partial F}{\partial \theta} - \frac{d}{dr}\frac{\partial F}{\partial \theta'} = 0 \tag{2}$$

(where $F(r, \theta, \theta')$ is the integrand in (1)) passing through the two given points makes (1) an *absolute* minimum,† and since in (1) F is independent

* In the American Math. Monthly, vol. 23 (1916), pp. 305–306, B. F. Finkel gives a proof in which, however, the shortest line is assumed from the outset to be a plane curve, and the differential equation of the problem is integrated in a rather complicated manner.

† Since (1) belongs to a general class of integrals shown to have this property by Carathéodory; see Bolza, Vorlesungen über Variationsrechnung, chapter IX.

of θ, (2) becomes $\dfrac{d}{dr}\dfrac{\partial F}{\partial \theta'} = 0$, whence $\dfrac{\partial F}{\partial \theta'} = \text{const.}$ or

$$\frac{r^2\theta'}{(R^2 - r^2)\sqrt{1 + r^2\theta'^2}} = \frac{1}{2\sqrt{a^2 - R^2}},$$

where a is a constant. Solving for θ',

$$\pm\,\theta' = \frac{R^2 - r^2}{r\sqrt{4a^2r^2 - (R^2 + r^2)^2}},$$

whence

$$\pm\, d\theta = \frac{\left(\dfrac{R^2}{r} - 1\right)dr}{\sqrt{4a^2 - \left(r + \dfrac{R^2}{r}\right)^2}} = -\frac{d\left(r + \dfrac{R^2}{r}\right)}{\sqrt{4a^2 - \left(r + \dfrac{R^2}{r}\right)^2}}$$

and integrating, α being an integration constant, we find

$$\pm\,(\theta - \alpha) = \operatorname{arc\,cos}\frac{1}{2a}\left(r + \frac{R^2}{r}\right),$$

or

(3) $$r^2 - 2\,ar\cos(\theta - \alpha) + R^2 = 0.$$

Writing this equation $r^2 + a^2 - 2\,ar\cos(\theta - \alpha) = a^2 - R^2$, the cosine theorem shows that it represents a circle with center at $r = a$, $\theta = \alpha$ and radius $\sqrt{a^2 - R^2}$, and since $a^2 = R^2 + (\sqrt{a^2 - R^2})^2$, this circle intersects the circle $r = R$ (and consequently the sphere) orthogonally.

Let ψ be the angle between the lines joining the center of the orthogonal circle to a point on it and to the center of the sphere; then

$$r^2 = a^2 + (\sqrt{a^2 - R^2})^2 - 2a\sqrt{a^2 - R^2}\cos\psi$$

and $ds = \sqrt{a^2 - R^2}\,d\psi$. Moreover, let ψ_1 and ψ_2 be the ψ-values corresponding to the two given points, and ψ_0 and $-\psi_0$, where $\sin\psi_0 = R/a$, those corresponding to the intersection points of the orthogonal circle and the sphere. Then

$$\int\frac{ds}{R^2 - r^2} = \frac{1}{2a}\int_{\psi_1}^{\psi_2}\frac{d\psi}{\cos\psi_0 - \cos\psi} = \frac{1}{2R}\log\frac{\sin\dfrac{\psi_1 + \psi_0}{2}\sin\dfrac{\psi_2 - \psi_0}{2}}{\sin\dfrac{\psi_1 - \psi_0}{2}\sin\dfrac{\psi_2 + \psi_0}{2}},$$

an expression familiar in non-euclidean metrics.